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# THE EQUAL ABSOLUTE SACRIFICE PRINCIPLE REVISITED

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**Abstract.** We summarize the literature on equal absolute sacrifice income taxes, and make some extensions. We adapt the utilitarian equal sacrifice criterion to a wide class of rank-dependent social welfare functions, and find that liabilities depend on both income *and* position in the distribution. We investigate whether such taxes need be progressive, using a combination of analytics and simulation, and in the process uncover tax functions not previously recognized as equating sacrifices. Finally, out of horizontal equity considerations a new concept of 'the equal treatment of equals' by an income tax emerges, with implications for future work whose significance is discussed.

**Keywords.** Equal sacrifice; Equity; Income tax

## 1. Introduction

In this paper, we consider the equal absolute sacrifice principle of income taxation, formulating this principle under several different assumptions about the nature of the underlying social welfare function (SWF). We then investigate an old question, whether such taxes need be progressive, under the different assumptions we have made about the SWF, using a combination of analytics and simulation to do this. The outcome is a thorough-going survey of the equal sacrifice literature to date, with an extension whose significance we shall discuss at the end of the paper.

The equal sacrifice principle was developed from the first of Adam Smith's (1776) four 'canons of taxation' that taxation should be equitable. Since the early work of Mill (1848), Cohen Stuart (1889), Carver (1895), Edgeworth (1897, 1925) and Pigou (1932), equal absolute sacrifice analysis has taken an intrinsically utilitarian form – that is, resting upon an assumed utility-of-income function – in terms of which all taxpaying income units experience the same absolute loss when the tax is imposed.[1] Yaari (1988) showed that the equal sacrifice principle can be articulated in terms of a rank-dependent and linear SWF, and in this case one's tax liability becomes a function of one's *position* in the distribution of income rather than one's income *per se*. However, Yaari's prescription has not been followed up in other literature. In this paper, we review what is known in each of these cases, and then we extend the equal sacrifice prescription to a class of 'hybrid' utilitarian and rank-dependent SWFs, which invoke a social utility-of-income function and

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also attribute weights giving systematically differing social importance to people's positions in the income distribution. Such SWFs can be traced back to Berrebi and Silber (1981), and to Quiggin's (1993) 'rank-dependent expected utility' evaluation function in the uncertainty literature. If such an SWF is invoked for distributional analysis then, we demonstrate, the equal sacrifice prescription leads to a tax function for which a person's liability is a function of *both* her income *and* her position in the income distribution.

The structure of the paper is as follows. In Section 2, we specify the relevant SWFs, explain their pedigrees, and consider the meaning of the equal sacrifice principle in each case. In Section 3, we examine the equal sacrifice tax functions for the various SWFs we have defined, and we trace out the analytical results concerning their progressiveness (or not) which are to be found in existing literature. Then we use simulation to illustrate the sort of shapes which the newly arising equal sacrifice tax functions can exhibit, uncovering in the process profiles not previously recognized as capable of representing equal sacrifices. Finally in this section, we discuss the horizontal equity (HE) characteristics of the tax schedules which arise for the distinct families of SWFs we have considered. In the concluding Section 4, we draw everything together, and suggest some new contexts in which the equal sacrifice prescription might be fruitfully brought to bear.

## 2. SWFs used in Distributive Analysis

### 2.1 *Utilitarian SWFs*

Perhaps the most familiar social evaluation function used in distributive analysis is the additively separable and symmetric form:

$$W_X = \frac{1}{N}\sum_{i=1}^{N} U(x_i) \tag{1a}$$

for an income distribution $X$ comprising incomes $x_1 \geq 0, x_2 \geq 0, \ldots, x_N \geq 0$, where $U$ is a social decision-maker's imposed utility-of-income function. If incomes are continuously distributed, with frequency density function $f(x)$, the analogous expression for per capita well-being is

$$W_X = \int_0^{\infty} U(x) f(x)\,\mathrm{d}x \tag{1b}$$

We shall use the discrete and continuous notations interchangeably in this paper. This classical SWF was popularized by Tony Atkinson's use of it in his 1970 paper 'On the measurement of inequality' which has been cited more than 1000 times in the following 37 years (see Lambert, 2008a, on this). The earlier work of Kolm (1969) anticipates this form, but see Kolm (1999, pp. 59–61) for thoughtful remarks about the meaningfulness of equations (1) as a measure of utilitarian well-being. One can also interpret equations (1) as a person's expected utility, measured from behind a 'veil of ignorance', which is specified in a thought experiment in such a way that the person may be identified with any one of the individuals populating the income distribution with the same probability. The probability of being person $i$ ($1 \leq i \leq N$) in the discrete case is $1/N$, and the probability of having an income in the small range $[x, x + \mathrm{d}x)$ in the continuous case is $f(x)\mathrm{d}x$. See Vickrey (1945) for the beginnings of this line of enquiry, Lerner (1944) for an even earlier and different probabilistic interpretation, and Lambert (2001, chapters 3 and 4) and Thistle (2007) for related discussion.

The utility function $U$ in equation (1) is usually assumed strictly increasing and continuous, and may or may not be concave (the latter property ensures egalitarianism on the part of the social evaluator or risk aversion from behind the veil of ignorance). A tax schedule $t(x)$, expressing the tax liability of a person with income $x$, is an equal absolute sacrifice tax for the utility function $U$ if, for all $x$ and some constant $c > 0$, $U(x) - U[x - t(x)] \equiv c$, but care is needed about the range of income values over which this identity is actually feasible. If $U(0) \neq -\infty$, the range of $x$ values must be bounded away from zero, else if we let $x \to 0$ in the identity, we would find that $t(0) > 0$, an impossible state of affairs. We therefore define an equal absolute sacrifice tax for the utility function $U$ by

$$U(x) - U[x - t(x)] = c \qquad \forall x \geq x_0 \tag{2}$$

where $x_0 \geq 0$ is some threshold income value, and the quantifier $\forall x \geq x_0$ can be read as 'for the incomes $x$ of all taxpayers', since only for such persons is $t(x) > 0$ as equation (2) demands.[2] The effect of this tax on welfare is to lower it by the amount $c$:

$$W_{X-T} = \frac{1}{N}\sum_{i=1}^{N} U[x_i - t(x_i)] = W_X - c \tag{3}$$

assuming that each income is taxable (and similarly in the continuous case).[3] One is thus seeking an income tax *such that the per capita welfare loss is experienced equally by all taxpayers.* This criterion will be our guide for other welfare functions yet to be introduced.

Before moving on, it is worth noting that, ever since the days of Robbins (1935) and Scitovsky (1951), interpersonal comparability of utilities has been an issue of concern among economists, and has been much researched in more modern times (see, for example, Blackorby *et al.*, 1984). The early proponents of equal sacrifice taxes clearly relied on the assumption of cardinal preferences, but Young (1988) has shown that the equal sacrifice approach can in fact be justified without appeal to utilitarianism. He shows that if a tax function $t(x)$ satisfies certain 'much more primitive concepts of distributive justice' (Young, 1988, p. 322) then a utility function $U$ exists relative to which $t(x)$ is an equal absolute sacrifice tax. An incentive-preserving schedule $t(x)$ in fact only needs to (a) be *strictly monotonic* and (b) satisfy the *composition principle* (two technical properties from allocation theory) for this result.[4] We shall return to the question of existence of a utility-of-income function relative to which a given tax structure delivers equal absolute sacrifices in Section 3.1 of the paper.

## 2.2 *Linear SWFs*

Yaari (1988) introduced a family of SWFs which are *linear* in peoples' incomes and also *rank dependent.* Associated with each such SWF is a measure which Yaari characterizes as the 'equality-mindedness' of the implied social preferences. Yaari formulated the equal sacrifice principle in the context of these SWFs, and showed how equality-mindedness forms an input to the determination of the equal absolute sacrifice tax function. In this section of the paper, we discuss Yaari's family of SWFs.

Let

$$Y_X = \frac{1}{N}\sum_{i=1}^{N} x_i \varphi'(p_i) \tag{4a}$$

where $\varphi(p)$, $0 \leq p \leq 1$, is an increasing and twice continuously differentiable function, and where $p_i$ is the rank of person $i$. If the income vector $X = (x_1, x_2, \ldots, x_N)$ is increasingly ordered, and if no two of the $x_i$ are the same, then $p_i = i/N$.[5] If incomes $x$ are continuously distributed with frequency density function $f(x)$ and distribution function $F(x) = \int_0^x f(t)\mathrm{d}t$, then we have

$$Y_X = \int_0^{\infty} x\varphi'(F(x))f(x)\mathrm{d}x \tag{4b}$$

Under the additional restrictions that $\varphi'(1) = 0$ and $\sum_{i=1}^{N} \varphi'(p_i) = 1$ or $\int_0^1 \varphi'(p)\mathrm{d}p = 1$, the evaluation function $Y_X$ in equations (4) defines Yaari's class of so-called *linear SWFs* – there is one for each such function $\varphi$.

In order that the policy-maker whose preferences are represented by $Y_X$ be 'equality-minded', Yaari requires $\varphi$ to be concave, $\varphi''(p) \leq 0\ \forall p \in [0, 1]$. In fact, he defines the 'local equality-mindedness' of the policy-maker as $-\varphi''(p)/\varphi'(p)$ at percentile $p \in [0, 1]$, and his aggregate 'equality rating' is $E_X = Y_X/\mu_X$ where $\mu_X$ is the mean of distribution $X$. Under Yaari's restrictions on $\varphi'(p)$, three desirable properties are gained: (i) a completely equal income distribution has an equality rating of 1, (ii) an extremely unequal distribution (in which one person has all of the money in the discrete case, or all of the density is at one income value in the continuous case) has an equality rating of 0, and (iii) $M_X = 1 - E_X$ is an index of relative inequality belonging to Mehran's (1976) 'general class of linear measures of income inequality'. In fact, every linear SWF is of the form

$$Y_X = \mu_X(1 - M_X) \tag{5}$$

where $M_X$ is a Mehran index and, conversely, every Mehran index $M_X$ defines a linear SWF, as $Y_X = \mu_X(1 - M_X)$.[6]

The best-known Mehran index is undoubtedly the Gini coefficient $G_X$. The associated linear SWF, $Y_X = \mu_X(1 - G_X)$, is obtained by setting $\varphi(p) = 2p - p^2$ in equation (4); this SWF was proposed by Sheshinski (1972) and popularized by Sen (1973). The extended Gini coefficient $G(\upsilon)$, where $\upsilon > 1$ is a distributional judgment parameter, is also becoming widely used. This inequality index is due to Weymark (1981), Donaldson and Weymark (1980, 1983) and Yitzhaki (1983), and features in the linear SWF $Y_X(\upsilon) = \mu_X[1 - G_X(\upsilon)]$, for which $\varphi_\upsilon(p) = 1 - (1 - p)^\upsilon$, the case $\upsilon = 2$ being that of the regular Gini coefficient.[7]

A tax schedule $t(x)$ that engenders equal sacrifices for all taxpayers, in terms of the linear SWF in equations (4), can be derived as follows. From equation (4a), each dollar taken in tax from person $i$ without disturbing the overall ranking of income units causes a loss of welfare of $(1/N)\varphi'(p_i)$. If $t(x)$ is incentive preserving, then there is no reranking of income units in the transition from pre- to post-tax income and we could imagine the taxes to be subtracted sequentially dollar by dollar from all persons while maintaining the ranking at every stage. Then person $i$ accounts for a loss of $(1/N)t(x_i)\varphi'(p_i)$ from pre-tax welfare, and the aggregate welfare loss is $(1/N)\sum_{i=1}^{N} t(x_i)\varphi'(p_i)$ per capita; call this $c > 0$. For equal sacrifices, we apparently require $t(x_i)\varphi'(p_i) = c\ \forall i$. However, there is a problem with this specification for the richest individual, since when $i = N$ we have $\varphi'(p_N) = 0$ by assumption. We must therefore restrict the equal sacrifice requirement to *exclude the individual(s) with the highest income level*, a significant difference from the utilitarian case.[8] (We do not, of course, rule out that the richest will pay tax, but their sacrifice will be different.) The equal sacrifice recipe now covers all income recipients except for the very richest, but may yet be inoperable at either extreme of the income distribution, or both, since admissibility on grounds of ability to pay ($t(x) \leq x\ \forall x$) is not guaranteed by our construction, and nor is incentive preservation, which we assumed in order to develop the recipe! We need to relax the recipe somewhat, and introduce upper and lower threshold income levels to delineate the region in which equal absolute sacrifices may be expected to hold in the case of a linear SWF. The equal sacrifice criterion is most conveniently expressed, and most easily analyzed, for the case in which income is continuously distributed:

$$t(x) = \frac{c}{\varphi'(F(x))} \quad \forall x \in [x_0, x_1] \tag{6}$$

for some $x_0 \geq 0$ and for some $x_1$ such that $F(x_1) \neq 1$. We must exclude extremely unequal income distributions from further consideration at this point: for such distributions, the quantifier '$\forall x$: $F(x) \neq 1$' simply means 'for $x = 0$'. Of course, as already remarked previously, we do not mean to suggest by equation (6) that upper income recipients should pay *no* tax – only that, beyond a certain point $x_1$ in the distribution, it may not be possible to equalize sacrifices. Other taxes may of course be applied in the extremes of an income distribution (see later). Notice that for linear SWFs a person's equal absolute sacrifice tax liability is a function of her income *x through her position F*($x$) *in the distribution in question*; it is not defined by her income *x per se*.

If an observed income tax function $t(x)$ is assumed to equalize absolute sacrifices for a linear SWF then, as Yaari has pointed out, information about the equality-mindedness of the policy-maker may be inferred from equation (6). He gives as an example the case of a lump-sum tax, $t(x) = \tau > 0\ \forall x$, assumed to equalize absolute sacrifices all along the income scale (except possibly at the very top). From equation (6), we have $\varphi'(p) = c/\tau\ \forall p \in (0, 1)$, and now from the property

$\int_0^1 \varphi'(p)\mathrm{d}p = 1$ we must have $c = \tau$. Thus, $\varphi'(p) \equiv 1\,\forall p < 1$, and zero equality-mindedness follows. Yaari explains this finding as follows: 'A policy maker who chooses such a tax policy (and finds it politically feasible) displays complete disregard for equality differences of income profiles. For such a policy maker, all income profiles will indeed have the same equality rating' (Yaari, 1988, p. 396).

### 2.3 *'Hybrid' Utilitarian and Rank-Dependent SWFs*

Yaari's class of linear SWFs grew out of his important contribution to the theory of risk, which was undergoing considerable development in the 1980s. See Yaari (1987). In this paper, Yaari sets up a 'dual' of expected utility theory, in which the roles of payoffs and probabilities are reversed; the dual evaluation function is linear in payoffs and concave in probabilities, rather than the other way round as for expected utility theory. If we think of income distributions $X$ as uncertain prospects being evaluated from behind a veil of ignorance, the utilitarian SWF of equations (1) and the linear SWF of equations (4) are similarly dual (compare equation (1b), $W_X = \int_0^\infty U(x)f(x)\mathrm{d}x$, with equation (4b), $Y_X = \int_0^\infty x\varphi'(F(x))f(x)\mathrm{d}x = \int_0^\infty x\varphi'(p)\mathrm{d}p$, where $p = F(x)$; in equation (1b) $U(x)$ is concave for inequality aversion and in equation (4b) $\varphi(p)$ is concave for equality-mindedness). A new form of risk theory, encapsulating both expected utility theory and its dual, grew out of the developments of the 1980s, and is laid out by Quiggin (1993). This is the so-called rank-dependent expected utility theory, whose evaluation function takes the form $Z = \int_0^\infty U(x)\varphi'(F(x))f(x)\mathrm{d}x$ with both $U(x)$ and $\varphi(p)$ concave (see Quiggin, 1993, chapters 5 and 14).

In income distributional analysis, a two-parameter sub-family of the rank-dependent expected utility class has in fact been known for some time. These are the 'hybrid' utilitarian and rank-dependent SWFs of the form

$$Z_X(e, \upsilon) = \frac{1}{N}\sum_{i=1}^{N} U_e(x_i)\varphi'_\upsilon(p_i) \tag{7a}$$

in the discrete case, and

$$Z_X(e, \upsilon) = \int_0^\infty U_e(x)\varphi'_\upsilon(F(x))f(x)\mathrm{d}x \tag{7b}$$

in the continuous case, which were introduced by Berrebi and Silber (1981) and further developed by Ebert (1988, pp. 155–156). In equations (7), the social utility-of-income function is $U_e(x) = x^{1-e}/(1-e),\ 0 \le e \ne 1,\ U_1(x) = \ln(x)$, which has constant relative inequality aversion $e \ge 0$ and is familiar from Atkinson (1970) as it defines the SWF he uses to underpin his celebrated inequality index; the weighting function $\varphi_\upsilon(p)$, which gives social importance to people's positions in the income distribution and underpins the linear SWF $Y_X(\upsilon) = \mu_X[1 - G_X(\upsilon)]$ featuring the extended Gini coefficient, has already been defined.

In combination, these Atkinson/extended Gini ingredients yield a class of SWFs whose parameters $e \ge 0$ and $\upsilon \ge 1$ are readily understood and can be varied to imbue the SWF – and its associated equal absolute sacrifice tax functions, as we shall see – with appropriate properties. The hybrid class has been studied closely by Araar and Duclos (2003, 2005). The special case $\upsilon = 1$ generates the utilitarian SWF based upon $U_e(x)$, and the special case $e = 0$ is that of Yaari's linear SWF. Taking $\upsilon = 2$ we have the 'Atkinson–Gini family' of SWFs invoked by Ebert and Welsch (2004) to study the preferences of a 'representative European individual' on the basis of happiness data in the Eurobarometer survey series.

$Z_X$ in equation (7) can equivalently be written as

$$Z_X(e, \upsilon) = \bar{W}_X(e)[1 - G(e, \upsilon)] \tag{8}$$

where $G(e, \upsilon) = G_{U_e(X)}(\upsilon)$ is the extended Gini coefficient defined over utility levels $U_e(x)$, $x \in X$, and $\bar{W}_X(e) = \int_0^\infty U_e(x) f(x) \mathrm{d}x$ is utilitarian welfare *à la* Atkinson (1970). In fact, $\bar{W}_X(e) = U_e\{\mu_X[1 - I_X(e)]\}$, where $I_X(e)$ is the Atkinson inequality index for the distribution $X$.[9] In terms of the form in equation (8), the special cases we have referred to are these: $Z_X(e, 1) = \bar{W}_X(e)$, $Z_X(0, \upsilon) = \mu_X[1 - G_X(\upsilon)]$ and $Z_X(e, 2) = \bar{W}_X(e)[1 - G(e, 2)] = U_e\{\mu_X[1 - I_X(e)]\}\,[1 - G(e, 2)]$ where $G(e, 2) = G_{U_e(X)}$ is the (regular) Gini coefficient of inequality in utility-of-income levels (hence $Z_X(e, 2)$ mixes the Atkinson (1970) form and the Gini coefficient, and is sometimes known as the Atkinson–Gini SWF).

What does an equal absolute sacrifice tax look like for one of these hybrid SWFs? If the tax function $t(x)$ is incentive preserving, each person $i$ accounts for a loss of $(1/N)\{U_e(x_i) - U_e[x_i - t(x_i)]\}\varphi'_\upsilon(p_i)$ from equation (7a) in the transition from pre- to post-tax welfare. Attending to possible problems at the two extremes of the income distribution, the equal absolute sacrifice tax function must satisfy

$$U_e(x) - U_e[x - t(x)] = \frac{c}{\upsilon[1 - F(x)]^{\upsilon - 1}} \quad \forall x \in [x_0, x_1] \tag{9}$$

where $c$ is the *per capita* sacrifice, for some $x_0 \geq 0$ and some $x_1$ such that $F(x_1) \neq 1$. Plainly the equal absolute sacrifice tax function depends on both $e$ and $\upsilon$. When $\upsilon = 1$, the equal sacrifice prescription in equation (9) reduces to equation (2) (for the utility function $U_e(x)$), and when $e = 0$, equation (9) reduces to equation (6) (for the weighting function $\varphi_\upsilon(p)$). In general, both a person's income *and* her position are ingredients in the determination of her tax liability for this class of SWFs.[10]

## 3. More about Equal Absolute Sacrifice Taxes for Utilitarian, Linear and Hybrid SWFs

### 3.1 *Utilitarian Equal Sacrifice Taxes*

What form of tax schedule $t(x)$ is compatible with the utilitarian equal absolute sacrifice principle, given a utility function $U(x)$? It is clear that if $U(x)$ is differentiable, then so is $t(x)$.[11] Differentiability of $U(x)$ therefore rules out a piecewise linear tax function. One can infer from equation (2) that whatever the utility function, the tax meets the ability to pay requirement ($0 < t(x) < x \; \forall x \geq x_0$)

and is incentive preserving ($0 \leq t'(x) < 1\ \forall x \geq x_0$). If $U(x)$ is linear, the tax is lump sum, and if $U(x)$ is logarithmic, it is proportional.[12] A tax is progressive if its average rate, $t(x)/x$, is weakly increasing in $x$, and strictly progressive if $t(x)/x$ is strictly increasing in $x$ (see Lambert, 2001, chapter 7, for discussion of the progressive principle). In the early literature, it was believed that equal absolute sacrifice justified only proportional or progressive taxation.[13] Samuelson (1947, p. 227) showed that if marginal utility is sufficiently declining as income rises, then equation (2) ensures strict progression,[14] but in fact regression is also compatible with equal absolute sacrifice: see later.

The question naturally arises, what sort of taxes can be rationalized as equal sacrifice taxes for *some* utility function $U(x)$? There are two strands of literature on this. One is theoretical, in which given properties of tax functions are identified with properties required of the associated utility functions, and existence theorems are developed. The other is empirical, seeking to fit actual tax systems exactly or approximately to the equal absolute sacrifice principle, using an appropriate utility function from a chosen parametric family.

Young (1987) discusses a 're-indexing property' of some tax schemes. This is where the tax $t(x)$ gets indexed to real growth and/or inflation, becoming $t^*(x) = Pt(x/P)$, where $P$ is the appropriate deflator. If, when $t(x)$ is an equal absolute sacrifice tax for some utility function $U(x)$, so is $t^*(x)$ (with maybe a higher or lower level of sacrifice, depending on the context), for the *same* utility function $U(x)$, then, Young shows, $U(x)$ must be a linear transformation of the utility function $U_e(x)$ already invoked to define the hybrid class of SWFs:

$$U_e(x) = x^{1-e}/(1-e), 0 \leq e \neq 1 \quad \text{and} \quad U_1(x) = \ln(x) \tag{10}$$

The equal absolute sacrifice tax function for $U_e(x)$ is proportional for $e = 1$ and takes the form

$$t(x) = x - [x^{1-e} - (1-e)c]^{1/(1-e)} \quad \forall x \geq x_0 \tag{11}$$

for $e \neq 1$.[15] It is strictly progressive if $e > 1$ and strictly regressive if $e < 1$ (above $x_0$). An index of 'progressivity', typically used to summarize the impact on inequality of a tax schedule given an income distribution, is that of Blackorby and Donaldson (1984):

$$\Pi^{BD}(e) = \frac{I_X(e) - I_{X-T}(e)}{1 - I_X(e)} \tag{12}$$

where $I_X(e)$ and $I_{X-T}(e)$ are the pre- and post-tax Atkinson (1970) inequality indices associated to the utility function $U_e(x)$. We have already encountered $I_X(e)$ within the pre-tax welfare measure $\bar{W}_X(e) = U_e\{\mu_X[1 - I_X(e)]\}$. If $g$ is the fraction of all income taken in tax (known as the 'total tax ratio'), then post-tax welfare is $\bar{W}_{X-T}(e) = U_e\{(1-g)\mu_X[1 - I_{X-T}(e)]\}$ analogously. Progressivity indices such as $\Pi^{BD}(e)$ depend on the original income distribution as well as the final one. They rate a strictly progressive tax positively, because such a tax unambiguously reduces overall inequality, and they rate a strictly regressive one negatively. For the equal absolute sacrifice tax function in equation (11), the welfare impact is

$$\bar{W}_X(e) - \bar{W}_{X-T}(e) = [1 - F(x_0)]c$$
$$= \bar{W}_X(e)\left\{1 - \left[(1-g)\left(1 + \Pi^{BD}(e)\right)\right]^{1-e}\right\} \quad (13a)$$

if $e \neq 1$ (where, of course, $\Pi^{BD}(e) \gtrless 0$ for $e \gtrless 1$), and

$$\bar{W}_X(e) - \bar{W}_{X-T}(e) = [1 - F(x_0)]c = -\ln\left\{(1-g)\left[1 + \Pi^{BD}(1)\right]\right\} \quad (13b)$$

for $e = 1$. These analytical results (which are new as far as we know) neglect any non-equal-sacrifice taxation which may occur below the threshold $x_0$, but see on. Clearly $c$, $g$ and $\Pi^{BD}(e)$ are all interdependent in equations (13). *Ceteris paribus*, the welfare reduction caused by the equal sacrifice tax is higher the greater the (required) yield, and lower the more progressive is the system. Also, if inequality aversion $e$ is raised *with tax revenue held constant*, then the new equal sacrifice tax function crosses the old one once, from below, and overall progressivity $\Pi^{BD}(e)$ rises.[16]

If the equal absolute sacrifice prescription operates only between the floor $x_0$ and some ceiling, $x_1$ say, and another form of tax is levied outside of this range, then of course the analytics leading to these redistributive properties is voided and ‘anything can happen’. In order to push the boundaries a little, but realistically, we used simulation. We drew 10,000 income values $x$ randomly from a lognormal distribution, the one for which $\ln x \sim \mathrm{N}(\theta, \sigma^2)$, where $\exp(\theta) = 33.818$ is median income and the variance of logarithms is $\sigma^2 = 0.154$. This lognormal distribution was found by Harrison (1981) to fit the UK distribution of gross weekly earnings in the year 1972. We also set $g = 0.15$, since the total tax ratio in the UK in 1972 was approximately 15% (Hutton and Lambert, 1980). Then we explored the associated equal absolute sacrifice income tax, *constraining the average tax rate $t(x)/x$ to lie between 5% and 50%* in order to retain a degree of realism in the results, and *continuing the tax at a 5% or 50% proportionate rate outside of the range indicated.* Figure 1 shows the tax functions we obtained for inequality aversion parameter values $e = {}^1/_2$, 1, $1{}^1/_2$ (these values are widely viewed to be reasonable; Atkinson (1970) used values between $e = {}^1/_2$ and $e = 2{}^1/_2$ to illustrate properties of his inequality index). In each panel of Figure 1 the tax level $t(x)$ is plotted as a heavy line and the average tax rate $t(x)/x$ is plotted as a pale line, against income $x$ measured as a fraction or multiple of the median.[17] In accord with the theory for an unrestricted equal sacrifice tax, this tax is regressive for $e = {}^1/_2$, proportional for $e = 1$, and progressive for $e = 1{}^1/_2$. Also, progressivity $\Pi^{BD}(e)$ increases as $e$ increases.

We continue by reporting the theoretical and empirical work which is to be found in the literature, seeking to match (typical properties of) actual tax functions with utility functions, enabling their interpretation as equal absolute sacrifice taxes.

In a powerful piece of analysis, Ok (1995) shows that any continuous and strictly increasing tax function $t(x)$ satisfying three innocuous properties, (i) $t(0) = 0$, (ii) $0 < t(x) < x \; \forall x > 0$, and (iii) the mapping $x \to x - t(x)$ is surjective, can in fact be rationalized as an equal absolute sacrifice tax for some strictly increasing

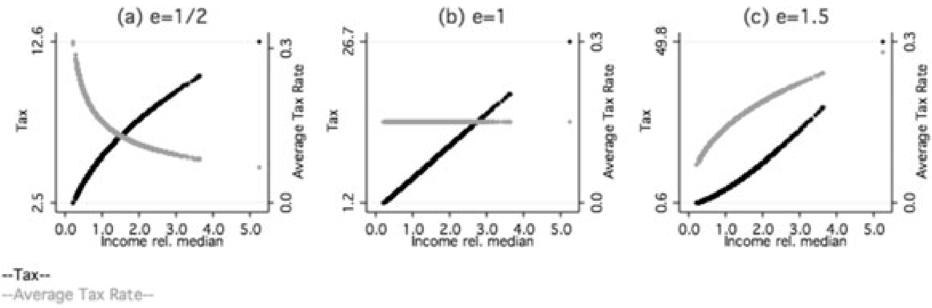



**Figure 1.** Utilitarian Equal Sacrifice Taxes for (a) $e = {}^1/_2$, (b) $e = 1$ and (c) $e = 1^1/_2$. The Progressivity Indices are (a) $\Pi^{BD}(^1/_2) = -0.0066$, (b) $\Pi^{BD}(1) = 0$ and (c) $\Pi^{BD}(1^1/_2) = 0.0159$.

and continuous utility function $U(x)$ (with $x_0$ set to zero in equation (2)), *if and only if $t(x)$ is incentive preserving*. He makes the point that the resulting utility function may be normatively unacceptable (for example if highly convex, which suggests risk-loving from behind the veil of ignorance and an aversion to rich-to-poor income transfers), and goes on to prove that a *convex* tax function $t(x)$ satisfying his postulates is an equal sacrifice tax for a *concave* utility $U$ under a mild additional restriction.[18] Mitra and Ok (1996) show that among piecewise-linear tax schedules, essentially *only* the convex ones are equal absolute sacrifice taxes if the utility function is required to be differentiable near the origin. D'Antoni (1999) extends Mitra and Ok's analysis to show that a two-bracket piecewise linear tax function, convex or not, is an equal sacrifice tax, and moreover with respect to an entire class of concave utility functions.[19] In Mitra and Ok (1997), the piecewise linearity assumption driving this result is dispensed with, and the authors show also that some non-convex progressive tax schedules are, and some are *not*, equal absolute sacrifice taxes for concave utility functions.

The equal sacrifice criterion has also been explored empirically. Mitra and Ok (1996) demonstrated that the statutory personal income tax codes in Turkey between 1981 and 1985, and in the USA between 1988 and 1990, though progressive, were *not* equal absolute sacrifice taxes for *any* concave utility function $U$. Young (1990) successfully modeled the US statutory income tax codes of 1957, 1967 and 1977 as equal absolute sacrifice taxes for isoelastic utility functions $U_e(x)$ noting, however, that 'at the lower and upper ends of the distribution, the ... [isoelastic] model does not fit the data well'. Gouveia and Strauss (1994) modeled *effective* US income taxes annually from 1979 to 1989 as equal absolute sacrifice taxes, also for the utility function $U_e(x)$.[20]

In this perusal of the utilitarian equal sacrifice literature, the quantifier $\forall x \geq x_0$ in equation (2) has played a role. In the theoretical work of Young (1987, 1988), Ok (1995) and Mitra and Ok (1996, 1997), equal sacrifice is held to apply universally, i.e. $x_0$ is set to zero throughout, forcing the restriction $U(0) = -\infty$ upon the respective models. Yet early writers such as Carver (1895) and Pigou (1932) were content to have the equal sacrifice principle applied over a range that is bounded away from zero, and Young (1990) found that his equal sacrifice model did not fit in the tails of the US income distribution. Once $x_0 > 0$ is allowed in equation (2), utility functions $U(x)$ are admitted into the analysis for which $U(0)$ is finite. Examples are the functions $U_e(x)$ for $e < 1$. The tax schedules corresponding to these, along with some of the schedules highlighted in the work of Mitra and Ok (1996, 1997) and D'Antoni (1999) for which $x_0 = 0$, are regressive (and one of them is shown in Figure 1(a)).

### 3.2 *Equal Sacrifice Taxes for the Class of Linear SWFs*

The equal absolute sacrifice tax for a linear SWF necessarily operates only below an upper cut-off income value $x_1$ and tax liability is determined by a person's position in the income distribution. These are significant differences from what pertains in the case of a utilitarian SWF.[21] Is the equal absolute sacrifice tax for a linear SWF progressive all along the income scale, and if not, does it anyway have positive overall progressivity? These questions are not easy to answer analytically, although we can make a little headway. Simulation will be used to add to the picture.

Neglecting possible other taxes outside of the range $[x_0, x_1]$, we can express the welfare loss occasioned by the equal sacrifice tax as

$$Y_X - Y_{X-T} = [F(x_1) - F(x_0)]c = Y_X[g - (1 - g)\Pi^M] \tag{14}$$

where $\Pi^M$ measures progressivity in terms of pre- and post-tax Mehran indices, *à la* Blackorby and Donaldson (1984):

$$\Pi^M = \frac{M_X - M_{X-T}}{1 - M_X} \tag{15}$$

Hence, as in the utilitarian case, *ceteris paribus* the welfare reduction is higher the greater the (required) tax yield, and lower the more progressive is the tax system. Clearly *c, g* and $\Pi^M$ are all interdependent in equation (14), and $\Pi^M < 0$ is possible.

Some information about the feasible range $[x_0, x_1]$ for equal sacrifices can be gleaned in the case of the SWF $Y_X(\upsilon) = \mu_X[1 - G_X(\upsilon)]$ involving the extended Gini coefficient. Using $\varphi_\upsilon(p)$ in equation (6), we have $t(x) = c/\upsilon[1 - F(x)]^{\upsilon-1}, x_0 \leq x \leq x_1$, where $x_0 > c/\upsilon$. Integrating,

$$\int_{x_0}^{x_1} t(x)f(x)\mathrm{d}x = g\mu_X = \frac{c}{\upsilon}\int_{F(x_0)}^{F(x_1)} \frac{\mathrm{d}p}{(1-p)^{\upsilon-1}}$$

This sets an upper limit upon $x_1$, since the integral becomes unbounded as $F(x_1) \to 1$, whereas $g \in [0, 1)$.

For further understanding, we turn to simulation. Figure 2 shows the equal absolute sacrifice tax functions which arise for a range of values of $\upsilon$ for the same lognormal income distribution as was used to construct Figure 1, with the total tax

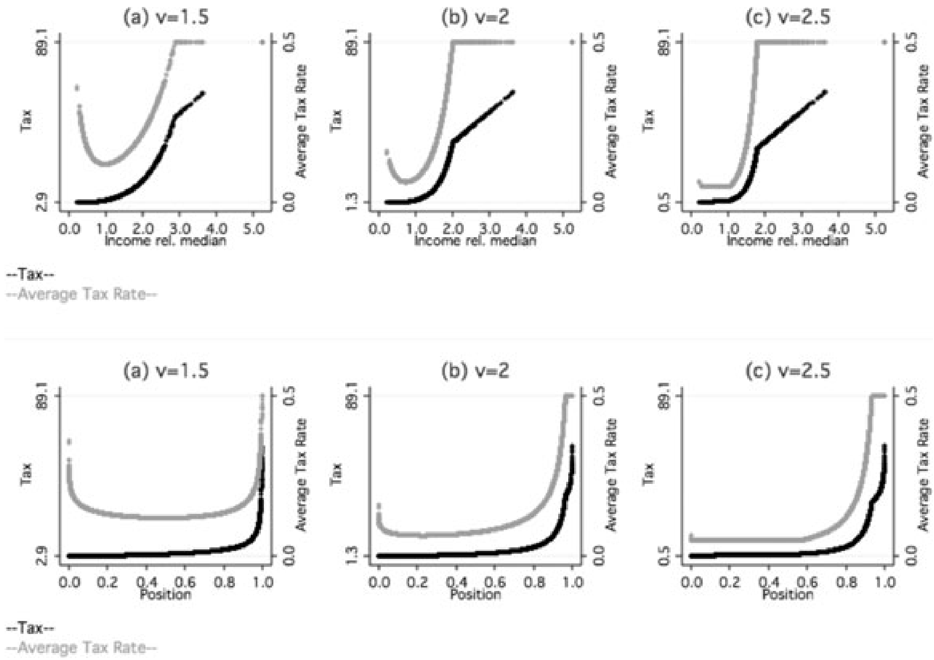



**Figure 2.** Equal Sacrifice Taxes for (a) $\upsilon = 1^1/_2$, (b) $\upsilon = 2$ and (c) $\upsilon = 2^1/_2$, as Functions of Income Relative to the Median and Position. The Progressivities are (a) $\Pi^M = 0.0166$, (b) $\Pi^M = 0.0720$ and (c) $\Pi^M = 0.0976$.

ratio again set at $g = 0.15$. Duclos (2000) recommends values of $\upsilon$ ranging from 1 to 4 for empirical work; we chose $\upsilon = 1^1/_2$, 2 and $2^1/_2$ for Table 1. In each panel of the top portion of Figure 2, the tax level (heavy) and average tax rate (pale) are shown against income relative to the median, and in the lower portion, these same taxes and average rates are shown as functions of position in the distribution. In all cases, we terminated the equal sacrifice prescription at income levels which kept the average tax rate within the 5%–50% range and, as for the utilitarian case, we continued the tax function outside of that range as proportional at the 5% or 50% flat rate. The regime changes thereby involved account for the non-differentiabilities in the tax function, which are observed at certain points in Figure 2.[22] The average tax rate is U-shaped in all cases, with the poorest income units experiencing high but declining average rates. As the lower portion of Figure 2 shows, the equal sacrifice prescription applies to more than 90% of the population for low $\upsilon$; as $\upsilon$ increases, the tax structure becomes 'highly polarized' (Yaari's (1987) words, p. 382), in that most taxpayers are driven to one or other extreme of the permitted range for the average rate. Overall progressivity $\Pi^M$ increases as $\upsilon$ increases.

3.3 *Equal Sacrifice Taxes for the Hybrid SWFs*

Using equation (9), the equal absolute sacrifice tax function for the hybrid SWF $Z_X(e, \upsilon)$ can be written explicitly as follows:

$$t(x) = x - \left[x^{1-e} - \frac{c(1-e)}{\upsilon[1-F(x)]^{\upsilon-1}}\right]^{1/(1-e)} \tag{16a}$$

if $e \neq 1$, and

$$t(x) = \left[1 - \exp\left\{\frac{-c}{\upsilon[1-F(x)]^{\upsilon-1}}\right\}\right] x \tag{16b}$$

if $e = 1$. When $e < 1$, the restriction

$$x_0[1-F(x_0)]^{(\upsilon-1)/(1-e)} > \left[\frac{c(1-e)}{\upsilon}\right]^{1/(1-e)} \tag{17}$$

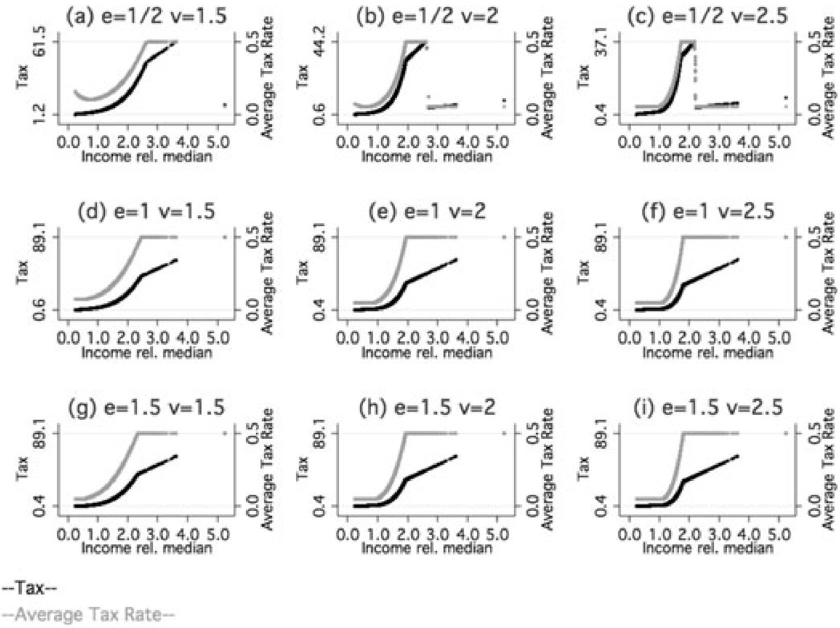
on $x_0$ is implied. Progressivity may be measured using the inequality index $G_{U_e(X)}(\upsilon)$ defined just below equation (8), as

$$\Pi^M(e, \upsilon) = \frac{G_{U_e(X)}(\upsilon) - G_{U_e(X-T)}(\upsilon)}{1 - G_{U_e(X)}(\upsilon)} \tag{18}$$

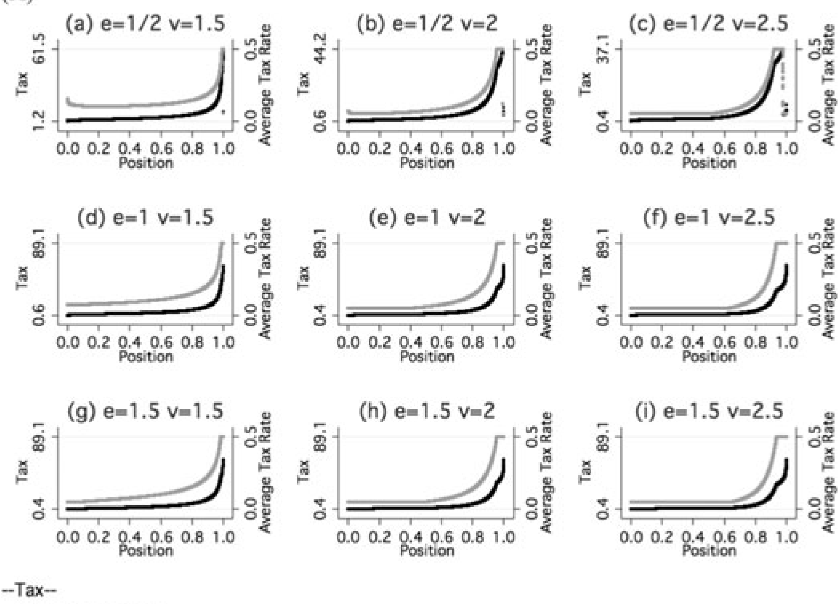
but the mathematics gets too complicated to go much further in the general case.

In Figure 3, we show the simulated tax functions for the cases $e = {}^1/_2$, 1, $1^1/_2$ and $\upsilon = 1^1/_2$, 2, $2^1/_2$ for the same lognormal income distribution as was used to construct Figures 1 and 2, and with the same total tax ratio $g = 0.15$. Figure 3(A) shows tax level (heavy) and average rate (pale) against income $x$ relative to the median and Figure 3(B) shows these same levels and average rates as functions of position $F(x)$.

It is striking that, as $e$ and $\upsilon$ increase, the interval between the two cut-offs $x_0$ and $x_1$ dictated by admissibility of the equal sacrifice prescription becomes smaller and smaller. The cut-offs $x_0$ and $x_1$ provide 'floors' and 'ceilings' for the average tax rate profiles but bring 'cusps' into the tax level/income relationships. Just as Yaari found for the linear case, the hybrid tax structure also becomes 'highly polarized' as $\upsilon$ increases. In the utilitarian setting (excluded here because $\upsilon > 1$ in all scenarios), the case $e = 1$ generated a proportional tax, and $e < 1$ a regressive tax. In the present case, for $e = 1$ the illustrated tax is progressive for all $\upsilon$, as it is for $e > 1$, whereas for $e = {}^1/_2$ and $\upsilon = 1^1/_2$ it is initially regressive (though for very few taxpayers indeed: see Figure 3(B)). Table 1 specifies the progressivity measures $\Pi^M(e, \upsilon)$ for the taxes illustrated in Figure 3. For $e = {}^1/_2$, we find that progressivity declines as $\upsilon$ becomes very large, but in general $\Pi^M(e, \upsilon)$ increases with both $e$ and $\upsilon$.




(A)




(B)

**Figure 3.** (A) Equal Absolute Sacrifice Tax Levels and Average Tax Rates for the Hybrid SWF in Terms of the Parameters $e$ and $\upsilon$, Shown as Functions of Income Relative to the Median. (B) Equal Absolute Tax Levels and Average Tax Rates for the Hybrid SWF in Terms of the Parameters $e$ and $\upsilon$, Shown as Functions of Position.

**Table 1.** Progressivity $\Pi^M(e, \upsilon)$ for the Hybrid SWF in Terms of the Parameters $e$ and $\upsilon$.

| | $\upsilon = 1^1/_2$ | $\upsilon = 2$ | $\upsilon = 2^1/_2$ |
|---|---|---|---|
| $e = {}^1/_2$ | 0.0348 | 0.0836 | 0.0944 |
| $e = 1$ | 0.0555 | 0.0963 | 0.1050 |
| $e = 1^1/_2$ | 0.0750 | 0.1019 | 0.1078 |

For some additional comparisons between equal absolute sacrifice tax functions for the utilitarian, linear and hybrid SWF classes, the reader may consult Lambert and Naughton (2006).[23]

### 3.4 *Horizontal Equity Considerations*

We turn finally to the issue of HE. HE calls for 'the equal treatment of equals' by a tax system. There are two issues to be addressed for HE: who are the equals, and what is meant by equal treatment? The equals are typically taken to be *those with the same pre-tax living standard*, and equal treatment typically means that these people *should have the same post-tax living standard*, but this is not universal.[24] HE in this form becomes a non-trivial question when a population is socially heterogeneous, and factors other than income $x$ alone are relevant for the determination of both living standards and tax liabilities (see Galbiati and Vertova, 2008, for a thoughtful essay). In this paper, we have – at least implicitly – been considering a socially homogeneous population, since our assumed utility-of-income functions are common to all persons (but see later for a discussion concerning the introduction of social heterogeneity). Then HE comes down simply to whether taxes are an incentive-preserving function of people's incomes – and they are for each of our equal absolute sacrifice tax functions. So what is the issue?

Because the linear and hybrid SWFs are *positional*, an interesting variation on the usual HE criterion is possible. Suppose we say that *the equals are those who make the same contribution to pre-tax social welfare*. Persons 1 and 2 will be equals for the utilitarian SWF in equation (1) if and only if $U(x_1) = U(x_2)$, i.e. if and only if $x_1 = x_2$, which is the usual criterion: nothing new arises here. But for the linear and hybrid SWFs, new HE criteria do arise. Persons 1 and 2 will be equals for the linear SWF if and only if $x_1\varphi'(F(x_1)) = x_2\varphi'(F(x_2))$, and for the hybrid SWF if and only if $U_e(x_1)[1 - F(x_1)]^{\upsilon-1} = U_e(x_2)[1 - F(x_2)]^{\upsilon-1}$. These conditions ensure that people with the same income are equals, but may also 'let in' as equals other income units with different incomes and positions. What property may we characterize as 'equal treatment', such that our equal absolute sacrifice tax functions treat equals equally? From equations (6) and (9), these tax functions satisfy $t(x_1)\varphi'(F(x_1)) = t(x_2)\varphi'(F(x_2))$ and $\{U_e(x_1) - U_e[x_1 - t(x_1)]\} \times (1 - F(x_1))^{\upsilon-1} = \{U_e(x_2) - U_e[x_2 - t(x_2)]\}(1 - F(x_2))^{\upsilon-1}$, respectively. Thus if persons 1 and 2 are equals, then $t(x_1)/x_1 = t(x_2)/x_2$ in the linear case, and $U_e[x_1 - t(x_1)]/U_e(x_1) = U_e[x_2 - t(x_2)]/U_e(x_2)$ in the hybrid case. In each case, *equals experience the same proportional utility sacrifice*.[25] This concept of equal treatment, and associated definition of equals, provides an appealing new HE criterion, which surely merits deeper study.

## 4. Conclusions and Ways Forward

This paper summarizes the existing literature on equal absolute sacrifice income taxes, and makes some extensions. Most of the existing literature has focused on the class of utilitarian SWFs. Yaari (1988), in contrast, discusses the equal absolute sacrifice principle in the framework of a linear SWF. While the properties of utilitarian equal sacrifice taxes are well-studied by now, the properties of equal sacrifice taxes for linear SWFs have not been investigated before now. An extended class of SWFs, which we have termed 'hybrid' because they both invoke a social utility-of-income function (as in the utilitarian case) and also attribute positional weights (as in the Yaari case), was formulated in the 1980s, but the equal sacrifice principle for such SWFs was not even articulated, let alone analyzed, in the relevant literature. Our paper aims to fill these gaps.

The utilitarian approach generates equal absolute sacrifice taxes that are progressive, flat or regressive at all levels of income. In realistic simulations, we find the equal absolute sacrifice tax for a linear SWF to be overall progressive, but to have an average rate profile that can be U-shaped. For the hybrid SWFs, the equal sacrifice taxes are shown, in our simulations, to be progressive for almost all taxpayers for the parameter configurations we explored.[26]

Equal absolute sacrifice taxes are generally held to obey the HE command, which calls for the equal treatment of equals by an income tax. In the case of linear and hybrid SWFs, the role of position as well as income in social evaluations leads to a possible new concept of 'who are the equals', and the taxes we have characterized as equalizing absolute sacrifices for these SWFs satisfy a new concept of HE. Namely, defining the equals as those who make the same contribution to pre-tax social welfare, the taxes we have outlined cause equals to experience the same *proportional* sacrifice *in* (*non-positional*) *utility terms*. This HE concept, brought up by our study, is certainly worthy of further investigation.

We conclude with some forward-looking remarks, first on the incentives and labor supply issue, which needs to be confronted in any study of the effects of taxation, and second, on the introduction of social heterogeneity into the equal absolute sacrifice model.

The classical equal absolute sacrifice literature did not take account of incentives, and nor has any of the subsequent literature which we have covered so far in this survey. However, Berliant and Gouveia (1993) have developed a model in which equal absolute sacrifice taxation is integrated with the more modern concept of incentive compatibility.[27] The authors see this extension as 'but a logical development of the normative theory of income taxation' (p. 221). They find that many of the standard results for utilitarian equal absolute sacrifice taxes carry over to their model, including one result which has deep significance for us: '*If utility* (now defined over consumption and leisure) *is additively separable for each consumer . . . an incentive compatible equal sacrifice tax function will induce the same level of labor supply (and gross income) as in the no tax case*' (Berliant and Gouveia, 1993, Proposition 4, p. 231). This result provides a rigorous justification of the classical results on utilitarian equal absolute sacrifice taxes: as the authors point out, 'we can ignore the incentives problem and proceed as if gross income were exogenous' (Berliant and Gouveia, 1993, p. 232). No such modeling has yet been developed for linear or hybrid SWFs, however. Such modeling would have to represent consumers' concerns for their ranks in the income distribution as well as their consumption and leisure bundles. A possible further extension might be to build in altruism or envy.[28]

The equal absolute sacrifice principle for utilitarian SWFs has not yet been adapted to cope with social heterogeneity, but the equal absolute sacrifice principle for hybrid SWFs has.

Atkinson and Bourguignon (1987) have reformulated the utilitarian SWF of equation (1) to allow for differences in need between socially homogeneous subgroups. The SWF remains additively separable, but is not symmetric because systematically differing utility-of-income functions $U^{[k]}(x)$ are invoked for the homogeneous subgroups $k = 1, 2, 3, \ldots$, to capture the needs structure.[29] It is an open question whether such SWFs, in conjunction with a variant of the equal absolute sacrifice principle, could explain the typical ways in which real-world income taxes allow for needs differences (using exemptions, deductions, credits, income splitting etc.).

The equal absolute sacrifice principle for hybrid SWFs has recently been adapted to the case of a socially heterogeneous population. In Lambert (2008b), some possible design principles for an EU-wide layer of income tax are discussed, prompted in part by a suggestion of the President of the European Parliament in 2006, that 'the direct taxation of the people would more closely involve individual citizens with the European government'. Under the assumption that country-specific net income distributions within the EU are all lognormal, and defining the equals in the different EU countries as those located at the same percentile points in these distributions, Lambert shows that under certain assumptions on the parameters of the assumed within-country SWFs $Z_{X_j}(e_j, \upsilon_j), j = 1, 2, 3, \ldots$, a set of tax schedules can be devised such that a person experiences the same absolute sacrifice as all of her country $j$ compatriots, and the same proportional sacrifice (in utility terms), as well as the same degree of income tax progression, as all of her peers (similarly placed individuals) in the other countries.

There is still plenty of life in the equal absolute sacrifice criterion, despite the topic having very old origins and the literature having fallen somewhat silent in recent years.

## Acknowledgments

We are grateful to Shlomo Yitzhaki, Peyton Young, Stuart Sayer, and two referees of this journal for encouragement.

## Notes

1. Mill (1848, Book V, chapter 2) argued for equal absolute sacrifice, whereas Cohen Stuart (1889) argued for the proportional version of the equal sacrifice principle. Carver (1895, pp. 96–97) considered both the marginal and absolute versions of the principle, and Edgeworth (1897) favored the marginal version. In the utilitarian case, the proportional version is merely the absolute version in disguise: a tax schedule $t(x)$ engenders equal proportional sacrifice for some utility function $V(x)$ if and only if the self-same schedule engenders equal absolute sacrifice for the utility function $U(x) = e^{V(x)}$. See Musgrave and Musgrave (1984, chapter 11) for a neat graphical depiction of equal absolute, proportional and marginal sacrifice tax functions. We shall have a little more to say about equal marginal sacrifice taxes later.
2. Carver saw that the poorest would in general have to be exempted from an equal sacrifice tax. Mill argued that certain forms of income (for example, subsistence needs and savings for retirement) should be exempted from tax. Pigou (1932, chapter 9) pointed out that an equal sacrifice tax on the better-off could finance transfers to the less well-off, also supporting the use of a bounded range in equation (2) to specify the equal sacrifice tax.
3. If there are incomes below $x_0$, equation (3) becomes $W_{X-T} = W_X - c[1 - F(x_0)]$, where $F(x_0)$ is the proportion of non-taxpayers in the population.
4. A tax schedule is said to be incentive preserving if post-tax income $x - t(x)$ increases with pre-tax income $x$, i.e. in the differentiable case, if $t'(x) < 1\ \forall x$. The two questions to which technical properties (a) and (b) provide affirmative answers are these, in plain English: 'If the total tax burden increases, does everyone pay more? Is the increase shared in a fair way?' (Young, 1988, p. 322). That these properties hold for an equal sacrifice tax $t(x)$ is easy to see: (a) if an increase in total revenue is required while retaining equal sacrifices, plainly everyone's tax liability must rise; and (b) if a second layer of tax $s(y)$ is levied on net incomes $y = x - t(x)$ to raise additional revenue, then the composite tax $s \circ t$ equalizes sacrifices if and only if the 'surtax' $s(y)$ does. Young's achievement is to obtain a converse result, that if generalized versions of (a) and (b) hold for a tax $t(x)$, then a utility function exists relative to which $t(x)$ equalizes sacrifices.
5. Assuming a domain in which no two incomes are equal is not essential though it simplifies the assignation of ranks, and also incidentally makes $Y_X$ differentiable in each $x_i$. Most of the analysis to follow will be demonstrated for the case in which incomes are continuously distributed.
6. All of these results are demonstrated mathematically in Lambert and Naughton (2006). Mehran indices take the general form $M_X = \int_0^1 [p - L_X(p)]k(p)\,\mathrm{d}p$, where $k(p) \geq 0$ is a function such that $\int_0^1 pk(p)\,\mathrm{d}p = 1$ and $L_X(p)$ is the Lorenz curve for $X$, so that if $p = F(y)$ then $L_X(p) = \int_0^y xf(x)\,\mathrm{d}x/\mu_X$. The close link between linear SWFs and linear inequality measures comes by setting $k(p) = -\varphi''(p)$ in the definition of $M_X$ or setting $\varphi'(p) = \int_p^1 k(q)\,\mathrm{d}q$ in the definition of $Y_X$.
7. The $k$ functions for the Gini and extended Gini coefficients are $k(p) \equiv 2\ \forall p$ and $k(p) = \upsilon(\upsilon - 1)(1 - p)^{\upsilon-2}$, respectively.
8. Another significant difference is that for linear SWFs the equal *proportional* and equal *absolute* sacrifice rules are very different (recall note 1). Since each person $i$ accounts for a welfare contribution of $x_i\varphi'(p_i)/N$ to $Y_X$ and a welfare loss of $t(x_i)\varphi'(p_i)/N$ as a result of taxation, only a proportional tax could engender an equal proportional sacrifice from all individuals in the Yaari framework.

9. The Atkinson index is defined for a continuous distribution as $I_X(e) = 1 - (1/\mu_X)(\int_0^\infty x^{1-e}\mathrm{d}x)^{1/(1-e)}$. It follows from this that $\int_0^\infty U_e(x)f(x)\,\mathrm{d}x = U_e\{\mu_X[1 - I_X(e)]\}$.
10. In fact, from equation (9), this applies whenever $(e, \upsilon) \neq (0, 1)$.
11. This is because $t(x) = x - U^{-1}(U(x) - c)$ from equation (2). For explanations of subsequent mathematical assertions, where these are omitted, the interested reader could consult Lambert and Naughton (2006).
12. From equation (2), if $U(x) = ax + b$, then $t(x) = c/a\ \forall x \geq x_0$ (from which $x_0 > c/a$ can be inferred), and if $U(x) = a\ln(x) + b$, then $t(x) = [1 - \exp(c/a)]x\ \forall x \geq x_0$ (and $x_0 = 0$ is feasible in this case).
13. Young (1987) cites Cohen Stuart (1889) and Edgeworth (1897) as already pointing out that this is a fallacy.
14. For a differentiable tax schedule, Samuelson's result is this: $-xU(x)/U'(x) > 1\ \forall x \geq x_0 \Rightarrow \mathrm{d}(t(x)/x)/\mathrm{d}x > 0\ \forall x \geq x_0$.
15. Perforce, $x_0 > [(1 - e)c]^{1/(1-e)}$ when $e < 1$. It is readily verified that if $t(x)$ satisfies equation (11) then for any $P > 0$, $t^*(x) = Pt(x/P)\quad \forall x \geq x_0^* = Px_0$ is also an equal sacrifice tax for $U_e(x)$, with sacrifice level $c^* = P^{1-e}c$.
16. See Lambert and Naughton (2006) on this, and Buchholz *et al.* (1988) and Moyes (2003) for more general consideration of the inequality-reducing properties of equal absolute sacrifice taxes.
17. We report income as a multiple of the median for ease of interpretation: the generated income values would be meaningless to most readers. The income distribution is based on 1972 weekly income in UK pounds.
18. This restriction is that $t(x)$ be differentiable in a neighborhood of $x = 0$, with $0 < t'(0) < 1$.
19. There is no incompatibility with Mitra and Ok's finding because the utility functions D'Antoni comes up with all have infinitely many points of non-differentiability near the origin.
20. Young found that the inequality aversion parameter values $e = 1.61$, $e = 1.52$ and $e = 1.72$ best fitted the 1957, 1967 and 1977 statutory codes in the USA. He also obtained fits for Germany in 1984, Italy in 1987 and Japan in 1987, with $e = 1.63$, $e = 1.40$ and $e = 1.59$ respectively, but neither the USA nor the UK provided a satisfactory fit in 1987. Gouveia and Strauss's best-fit values of $e$ for the USA lay between 1.72 and 1.94. Creedy (2006, pp. 13–14) warns against using estimates of $e$, inferred from an observed tax system, as assumed value judgments in order to evaluate changes in tax policy.
21. Recall, though, that Young's (1990) utilitarian equal sacrifice tax did not fit in the upper tail of the US income distribution.
22. If we had confined the tax to the range $[x_0, x_1]$, rather than extending it beyond, then a higher sacrifice and tax level for all incomes would have been necessitated in order to meet the fixed yield requirement.
23. *Inter alia*, it is shown in Lambert and Naughton (2006) that the individual sacrifice level for a utilitarian tax, *as judged by a social decision-maker with a linear SWF*, is maximal at around median income; and that the Gini-based hybrid equal sacrifice taxes differ markedly from the corresponding utilitarian ones for low values of inequality aversion $e$, but become quite similar as inequality aversion is increased.
24. See Lambert (2001, chapter 10) for a survey.
25. These properties are obtained by division, for example of $t(x_1)\varphi'(F(x_1)) = t(x_2)\varphi'(F(x_2))$ by $x_1\varphi'(F(x_1)) = x_2\varphi'(F(x_2))$. We can consider utility to be defined as income $x$ for the linear SWFs.

26. Of course, if $\upsilon = 1$ the hybrid SWF is utilitarian, and then for $e < 1$ the equal absolute sacrifice tax is everywhere regressive.
27. See Laffont (1988) on incentive compatibility.
28. Berliant and Gouveia also discuss optimal income tax theory in their paper. In this theory, introduced by Mirrlees (1971), a planner maximizes an SWF subject to the technical constraints of the economy and the incentive constraint. Berliant and Gouveia view the equal sacrifice approach as 'a useful alternative or complement to optimal income tax theory ... [with] advantages ... in terms of tractability, simplicity, empirical content, and the transparency of its normative criterion' (Berliant and Gouveia, 1993, pp. 223–224). They suggest that equal *marginal* sacrifice taxation, which seeks to equate the after-tax marginal utilities of all taxpayers, could in principle be developed further using optimal tax theory. In respect of envy and altruism, Oswald (1983) has shown that '... all of the general results of optimal tax theory can be overturned by the plausible assumption that people look over their shoulders before they decide how happy they feel' (p. 86).
29. The main one is that the marginal utilities $U^{[k]\prime}(x)$, expressing the social benefit from an additional dollar, are ranked at all income values: $U^{[k]\prime}(x) > U^{[k+1]\prime}(x)\ \forall x$. The utilities $U^{[k]}(x)$ are concave, and the difference in social benefit for adjacent types, $U^{[k]\prime}(x) - U^{[k+1]\prime}(x)$, decreases as income $x$ rises. See Lambert (2001, chapter 4.4) for more on this.

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